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A bipartite strengthening of the Crossing Lemma ☆

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A B S T R A C T

Let $G=(V,E)$ be a graph with n vertices and $m \geqslant 4n$ edges drawn in the plane. The celebrated Crossing Lemma states that G has at least $\Omega(m^3/n^2)$ pairs of crossing edges; or equivalently, there is an edge that crosses $\Omega(m^2/n^2)$ other edges. We strengthen the Crossing Lemma for drawings in which any two edges cross in at most $O(1)$ points. An ℓ*-grid* in the drawing of G is a pair $E_1, E_2 \subset E$ of disjoint edge subsets each of size ℓ such that every edge in E_1 intersects every edge in E_2. If every pair of edges of G intersect in at most k points, then G contains an ℓ-grid with $\ell \geqslant c_k m^2/n^2$, where $c_k > 0$ only depends on k. Without any assumption on the number of points in which edges cross, we prove that G contains an ℓ-grid with $\ell = m^2/n^2 \operatorname{polylog}(m/n)$. If G is dense, that is, $m = \Theta(n^2)$, our proof demonstrates that G contains an ℓ-grid with $\ell = \Omega(n^2/\log n)$. We show that this bound is best possible up to a constant factor by constructing a drawing of the complete bipartite graph $K_{n,n}$ using expander graphs in which the largest ℓ-grid satisfies $\ell = \Theta(n^2/\log n)$.

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1. Introduction

A *drawing* of a graph[3] G in the plane is an embedding of the vertices to distinct points in the plane and a mapping of the edges to simple continuous arcs (for short, *curves*) connecting the corresponding vertices, but not passing through any other vertex. A *crossing* is a pair of curves and a common interior point between the two arcs (intersections at vertices do not count as crossings). The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings in a drawing of G. A celebrated result of Ajtai et al. [1] and Leighton [14], known as *the Crossing Lemma*, states that the crossing number of every graph G with n vertices and $m \geqslant 4n$ edges satisfies

$$\mathrm{cr}(G) = \Omega\left(\frac{m^3}{n^2}\right). \tag{1}$$

The currently best known constant coefficient is due to Pach et al. [18]. Leighton [14] was motivated by applications to VLSI design. Székely [28] used the Crossing Lemma to give simple proofs of the Szemerédi–Trotter bound on the number of point-line incidences [29], a bound on Erdős's unit distance problem, and a bound on Erdős's distinct distance problem [5]. The Crossing Lemma has since found many important applications, in combinatorial geometry [3,12,20,23,26,27], and number theory [4,30].

The *pairwise crossing number* pair-cr(G) of a graph G is the minimum number of pairs of crossing edges in a drawing of G [24]. The lower bound (1) also holds for the pairwise crossing number with the same proof. It follows that in every drawing of a graph with n vertices and $m \geqslant 4n$ edges, there is an edge that crosses at least $\Omega(m^2/n^2)$ other edges. Conversely, if in every drawing of every graph with $m \geqslant 3n$ edges some edge crosses $\Omega(m^2/n^2)$ others, then we have pair-cr$(G) = \Omega(m^3/n^2)$ for every graph G with $m \geqslant 4n$ edges. Indeed, by successively removing edges that cross many other edges, we obtain the desired lower bound for the total number of crossing pairs.

Definition 1.1. An *ℓ-grid* in a drawing of a graph $G = (V, E)$ is a pair $E_1, E_2 \subset E$ of disjoint edge subsets each of size ℓ such that every edge in E_1 crosses every edge in E_2.

Drawing with a bounded number of crossings between any two edges. We prove a bipartite strengthening of the Crossing Lemma for drawings where any two edges cross in at most a constant number of points by showing that such drawings contain a large ℓ-grid.

Theorem 1.2. *For every $k \in \mathbb{N}$, there is a constant $c_k > 0$ such that every drawing of a graph $G = (V, E)$ with n vertices and $m \geqslant 3n$ edges in which no two edges cross in more than k points contains an ℓ-grid with $\ell \geqslant c_k m^2/n^2$.*

We have $k = 1$ in straight-line drawings, $k = (t+1)^2$ if every edge is a polyline with up to t *bends*, and $k = d^2$ if the edges are sufficiently generic algebraic curves (e.g., splines) of degree at most d. Note also that every graph G has a drawing with $\mathrm{cr}(G)$ crossings in which any two edges cross at most once [31].

Previously, Pach and Solymosi [21] proved Theorem 1.2 in the special case of straight-line drawings of dense graphs. Later, Pach et al. [17] proved that for every $\ell \in \mathbb{N}$, every drawing of a graph with n vertices and at least $(8 \cdot 24^{\ell}\ell)n$ edges, where any two edges cross at most once, contains an ℓ-grid. Theorem 1.2 improves the upper bound on the minimum number of edges that guarantee an ℓ-grid in any drawing of the graph where any two edges cross at most once from $(8 \cdot 24^{\ell}\ell)n$ to $(c\sqrt{\ell})n$, where c is an absolute constant, and this bound is tight apart from the constant factor c.

General drawings. The dependence on k in Theorem 1.2 is necessary. We show that one cannot expect to find an ℓ-grid of size $\ell = \Omega(m^2/n^2)$ in a drawing if any two edges may cross arbitrarily many

[3] The graphs considered here are simple, having no loops or parallel edges.

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times, even if the graph drawings are restricted to be x-monotone. An *x-monotone curve* is a curve that intersects every vertical line in at most one point. A drawing of a graph is *x-monotone* if every edge is mapped to an x-monotone curve.

Theorem 1.3. *For every positive integer n, there is an x-monotone drawing of the complete bipartite graph $K_{n,n}$ such that $\ell = O(n^2/\log n)$ for every ℓ-grid in the drawing.*

It is not difficult to show that Theorem 1.3 implies that for every $n, m \in \mathbb{N}$ with $m \leqslant n^2/4$, there is a bipartite graph $G = (V, E)$ with n vertices, m edges, and an x-monotone drawing such that every ℓ-grid in this drawing satisfies $\ell = O(\frac{m^2}{n^2 \log(m/n)})$. Indeed, we may take G to be the disjoint union of appropriately chosen dense bipartite graphs. We can take $n^2/(4m)$ copies of $K_{2m/n,2m/n}$, for instance, if n and m are powers of 2, and this construction can be adjusted for all other possible values of $n, m \in \mathbb{N}$.

If any two edges may cross arbitrarily many times, we have the following bounds.

Theorem 1.4.

(i) *Every drawing of a dense graph $G = (V, E)$ with n vertices and $m = \Theta(n^2)$ edges contains an ℓ-grid with $\ell = \Omega(n^2/\log n)$.*
(ii) *There is a constant c such that every drawing of a graph $G = (V, E)$ with n vertices and $m \geqslant 3n$ edges contains an ℓ-grid with $\ell \geqslant \frac{m^2}{n^2 \log^c(m/n)}$.*

Theorem 1.3 shows that Theorem 1.4(i) for dense graphs is tight up to a constant factor; and Theorem 1.4(ii) is tight up to the exponent c of the polylogarithmic factor.

Pach et al. [17] also proved that for every $\ell \in \mathbb{N}$, every drawing of a graph with n vertices and at least $(16 \cdot 24^{4^\ell}\ell)n$ edges contains an ℓ-grid. Theorem 1.4 improves the upper bound on the minimum number of edges that guarantee an ℓ-grid in any drawing of the graph from $(16 \cdot 24^{4^\ell}\ell)n$ to $(\sqrt{\ell \log^c \ell})n$, where c is an absolute constant, and this bound is tight apart from the constant c in the exponent of the logarithmic factor.

Organization. We prove Theorem 1.2 in Section 2. We discuss how to modify the proof of Theorem 1.2 to obtain Theorem 1.4 in Section 3. We use expander graphs to construct an x-monotone drawing of $K_{n,n}$ with no large ℓ-grid in Section 4. Finally, we present a further strengthening of the Crossing Lemma for graphs satisfying some monotone property in Section 5.

2. Proof of Theorem 1.2

Intersection patterns of curves. The proof of Theorem 1.2 relies on a recent result on the intersection pattern of curves in which no two curves intersect in more than k points. For a collection C of curves in the plane, the *intersection graph* $G(C)$ is defined on the vertex set C, two elements of C are *adjacent* iff the (relative) interiors of the corresponding curves intersect. A complete bipartite graph is *balanced* if the vertex classes differ in size by at most one. For brevity, we call a balanced complete bipartite graph a *bi-clique*.

Theorem 2.1. *(See [9].) Given m curves in the plane such that at least εm^2 pairs intersect and any two curves intersect in at most k points for some $k \in \mathbb{N}$ and $\varepsilon > 0$, the intersection graph of the curves contains a bi-clique with at least $a_k \varepsilon^{64} m$ vertices where $a_k > 0$ depends only on k.*

If follows from the Crossing Lemma that in every drawing of a dense graph, the intersection graph of the edges is also dense. Therefore, Theorem 2.1 implies Theorem 1.2 in the special case that G is dense. This connection was first observed by Pach and Solymosi [21], in the case of straight-line drawings.

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If a graph G is *not* dense, we find an induced subgraph H for which the intersection graph of the edges of H is so dense that Theorem 2.1 guarantees an ℓ-grid already in H for an ℓ claimed by Theorem 1.2. We search for such an induced subgraph H using an algorithm (Algorithm 2.3 below) reminiscent of [22] that decomposes G recursively into induced subgraphs. The decomposition algorithm successively removes *bisectors*, and we use Theorem 2.2 below to keep the total number of deleted edges under control.

The *bisection width*, denoted by $b(G)$, is defined for every simple graph G with at least two vertices. It is the smallest nonnegative integer such that there is a partition of the vertex set $V = V_1 \cup^* V_2$ with $\frac{1}{3} \cdot |V| \leqslant V_i \leqslant \frac{2}{3} \cdot |V|$ for $i = 1, 2$, and $|E(V_1, V_2)| = b(G)$. Pach, Shahrokhi and Szegedy [19] gave an upper bound on the bisection width in terms of the crossing number and the L_2-norm of the degree vector (it is an easy consequence of the weighted version of the famous Lipton–Tarjan separator theorem [15,10]).

Theorem 2.2. *(See [19].) Let G be a graph with n vertices of degree $d_1, d_2, \ldots, d_n$. Then*

$$b(G) \leqslant 10\sqrt{\mathrm{cr}(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2(G)}. \tag{2}$$

Preprocessing. Let D be a drawing of a graph G. Since we want to keep the sum of degree squares under control, we preprocess the graph and its drawing to cap the maximum degree, while keeping the intersection graph of the edges intact. We transform the drawing D into a drawing D' of a graph $G' = (V', E')$ with m edges, at most $2n$ vertices, and maximum degree at most $2m/n$, so that the intersection graph of E' is isomorphic to that of E. If the degree d of a vertex $v \in V$ is above the average degree $\bar{d} = 2m/n$, split v into $\lceil d/\bar{d} \rceil$ vertices $v_1, \ldots, v_{\lceil d/\bar{d} \rceil}$ arranged along a circle of small radius centered at v. Denote the edges of G incident to v by $(v, w_1), \ldots, (v, w_d)$ in clockwise order in the drawing D. In G', connect w_j with v_i if and only if $\bar{d}(i-1) < j \leqslant \bar{d}i$, where $1 \leqslant j \leqslant d$ and $1 \leqslant i \leqslant \lceil d/\bar{d} \rceil$. Two edges of G' cross if and only if the corresponding edges of G cross. Also, letting $d(v)$ denote the degree of vertex v in G', the number of vertices of G' is

$$\sum_{v \in V} \lceil d(v)/\bar{d} \rceil < \sum_{v \in V} 1 + d(v)/\bar{d} = 2n.$$

Decomposition algorithm. We search for a sufficiently dense subgraph of G' using an algorithm reminiscent of [22] that decomposes G' recursively into induced subgraphs. In each step, the largest subgraphs H are split into two induced subgraphs of roughly equal size by removing a bisector of size $b(H)$.

Algorithm 2.3. DECOMPOSE(G')

1. *Let $S_0 = \{G'\}$ and $i = 0$.*
2. *While $(3/2)^i < 4n^2/m$, do*
   Set $i := i + 1$. Let $S_i := \emptyset$. For every $H \in S_{i-1}$, do
   - *If $|V(H)| \leqslant (2/3)^i\, 2n$, then let $S_i := S_i \cup \{H\}$;*
   - *otherwise split H into induced subgraphs H_1 and H_2 along a bisector of size $b(H)$, and let $S_i := S_i \cup \{H_1, H_2\}$.*
3. *Return S_i.*

For every i, every graph $H \in S_i$ has at most $|V(H)| \leqslant (2/3)^i\ 2n$ vertices. Hence, the algorithm terminates in $t = \lceil \log_{(3/2)}(4n^2/m) \rceil$ rounds, and it returns a set S_t of induced subgraphs, each of which has at most $(2/3)^t 2n \leqslant 2n/(4n^2/m) = m/2n$ vertices.

We introduce some more notation for the analysis of Algorithm 2.3. Let $T_i \subset S_i$ be the set of those graphs in S_i that have more than $(2/3)^{i+1}\ 2n$ vertices. Notice that $|T_i| \leqslant (3/2)^{i+1}$. Denote by G_i the

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disjoint union of the induced subgraphs in S_i. For an induced subgraph H of G', let $L_2(H)$ denote the square root of the sum of degree squares in H. A few immediate observations are in place.

Proposition 2.4. *Let G' be a graph with m edges and at most $2n$ vertices. Algorithm 2.3 deletes more than $m/2$ edges of G'.*

Proof. G' has at most $2n$ vertices. Each vertex lies in an induced subgraph in S_t containing at most $m/2n$ vertices. Hence, the total number of vertex pairs lying in a same induced subgraph of S_t is less than $\frac{1}{2}\cdot 2n\cdot(m/2n)=m/2$, and so the decomposition algorithm has deleted more than $m/2$ edges. □

Proposition 2.5. *Let $G'=(V',E')$ be a graph with at least n but at most $2n$ vertices, $m\geqslant 3n$ edges, and maximum degree at most $\bar{d}=2m/n$. The ith round of Algorithm 2.3 partitions every induced subgraph in T_i, and we have*

$$\sum_{H\in T_i} L_2(H)\leqslant \frac{2m}{\sqrt{n}}\sqrt{(3/2)^{i+1}}. \tag{3}$$

Proof. Denoting by $d(v,H)$ the degree of vertex v in an induced subgraph H, we have

$$\begin{aligned}\sum_{H\in T_i} L_2(H)=\sum_{H\in T_i}\sqrt{\sum_{v\in V(H)} d^2(v,H)}&\leqslant\sqrt{|T_i|}\sqrt{\sum_{v\in V(G_i)} d^2(v,G_i)}\\ &\leqslant\sqrt{(3/2)^{i+1}}\sqrt{n\cdot(\bar{d})^2}\leqslant\frac{2m}{\sqrt{n}}\sqrt{(3/2)^{i+1}}.\end{aligned} \tag{4}$$

In the first inequality, we use the Cauchy–Schwarz inequality to get $\sum_{H\in T_i}\sqrt{x_H}\leqslant\sqrt{|T_i|}\sqrt{\sum_{H\in T_i}x_H}$ with $x_H=\sum_{v\in V(H)}d^2(v,H)$. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $G=(V,E)$ be a graph with n vertices and $m\geqslant 3n$ edges. Since a graph with more than $3n-6$ edges cannot be planar, it must have a pair of crossing edges. Hence, as long as $3n\leqslant m<10^5n$, Theorem 1.2 holds as G contains a 1-grid and $1\geqslant 10^{-10}m^2/n^2$. We assume $m\geqslant 10^5n$ in the remainder of the proof.

Preprocess graph G and its drawing D as described above to obtain a graph $G'=(V',E')$ with at most $2n$ vertices, m edges, and maximum degree at most $2m/n$ such that the intersection graph of E' is isomorphic to that of E. For an induced subgraph H of G', let $p(H)$ denote the number of pairs of crossing edges in H in the drawing D', and let $e(H)$ be the number of edges of H. Theorem 2.1 implies that the intersection graph of the edges of an induced subgraph H of G' contains a bi-clique of size at least $a_k(\frac{p(H)}{e(H)^2})^{64}e(H)$, where $a_k>0$ is the constant depending on k only in Theorem 2.1. This further implies Theorem 1.2 for G' (and hence for G) if

$$\varepsilon_k\frac{m^2}{n^2}\leqslant\left(\frac{p(H)}{e(H)^2}\right)^{64}e(H), \tag{5}$$

where $\varepsilon_k>0$ is any constant depending on k only. We use $\varepsilon_k=(10^{10}k)^{-64}$ for convenience. Hence, it is enough to find an induced subgraph H for which

$$\frac{e(H)^{2-1/64}}{10^{10}}\left(\frac{m}{n}\right)^{\frac{1}{32}}<kp(H), \tag{6}$$

since this readily implies (5).

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Next, we decompose the graph G' with Algorithm 2.3. We show that one of the induced subgraphs H in the algorithm satisfies (6), otherwise the algorithm cannot delete more than $m/2$ edges, contradicting Proposition 2.4. We use Theorem 2.2 for estimating the number of edges deleted throughout the decomposition algorithm.

Assume, to the contrary, that (6) does not hold for any induced subgraph H of G'. This gives an upper bound on $kp(H)$. Note that $\mathrm{cr}(H) \leqslant kp(H)$ since any two edges cross in at most k points in the drawing D'. Substituting the upper bound for $kp(H)$ and using Jensen's inequality for the concave function $f(x) = x^{1-1/128}$, we have for every $i = 0, 1, \dots, t-1$,

$$\sum_{H \in T_i} \sqrt{\mathrm{cr}(H)} \leqslant \sum_{H \in T_i} \sqrt{kp(H)} \leqslant \sum_{H \in T_i} \sqrt{\frac{e(H)^{2-1/64}}{10^{10}} \left(\frac{m}{n}\right)^{\frac{1}{32}}} = 10^{-5} \left(\frac{m}{n}\right)^{\frac{1}{64}} \sum_{H \in T_i} e(H)^{1-\frac{1}{128}}$$
$$\leqslant 10^{-5} \left(\frac{m}{n}\right)^{\frac{1}{64}} |T_i|^{\frac{1}{128}} m^{1-\frac{1}{128}} \leqslant 10^{-5} \left(\frac{3}{2}\right)^{\frac{i+1}{128}} \frac{m^{1+1/128}}{n^{1/64}}. \tag{7}$$

By Theorem 2.2, the total number of edges deleted during Algorithm 2.3 is

$$\sum_{i=0}^{t-1} \sum_{H \in T_i} b(H) \leqslant 10 \sum_{i=0}^{t-1} \sum_{H \in T_i} \sqrt{\mathrm{cr}(H)} + 2 \sum_{i=0}^{t-1} \sum_{H \in T_i} L_2(H)$$
$$\leqslant 10^{-4} \frac{m^{1+1/128}}{n^{1/64}} \sum_{i=0}^{t-1} (3/2)^{\frac{i+1}{128}} + 4 \frac{m}{\sqrt{n}} \sum_{i=0}^{t-1} \sqrt{(3/2)^{i+1}}$$
$$< 10^{-4} \frac{m^{1+1/128}}{n^{1/64}} \left(\frac{4n^2}{m}\right)^{\frac{1}{128}} \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{\frac{i+1}{128}} + 4 \frac{m}{\sqrt{n}} \sqrt{\frac{4n^2}{m}} \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{\frac{i+1}{2}}$$
$$< \frac{1}{10} \cdot \frac{m^{1+1/128}}{n^{1/64}} \left(\frac{n^2}{m}\right)^{1/128} + 50 \cdot m^{1/2} n^{1/2} < \frac{m}{2}.$$

The second inequality uses inequalities (3) and (7); the third inequality uses the geometric series formula and the upper bound $t \leqslant 1 + \log_{(3/2)}(4n^2/m)$; while the last inequality follows from the fact that $m \geqslant 10^5 n$.

So at least $m/2$ edges survive, contradicting Proposition 2.4. We conclude that an induced subgraph H of G' satisfies (6). This completes the proof of Theorem 1.2. □

3. General drawings

A *string graph* is an intersection graph of a collection of curves in the plane. The *incomparability graph* of a partially ordered set $(P, \prec)$ has vertex set P and two elements of P are adjacent if and only if they are incomparable by $\prec$. Golumbic, Rotem and Urrutia [11] showed that every incomparability graph is a string graph. The proof is discussed in the next section and is motivation for the proof of Theorem 1.3. The following theorem implies that every dense string graph contains a dense subgraph which is an incomparability graph.

Theorem 3.1. *(See [7].) There is a constant c such that for every collection C of m curves in the plane whose intersection graph has εm^2 edges, we can pick for each curve $\gamma \in C$ a subcurve γ' such that the intersection graph of $\{\gamma' \colon \gamma \in C\}$ has at least $\varepsilon^c m^2$ edges and is an incomparability graph. In particular, every string graph on n vertices and εm^2 edges has a subgraph with at least $\varepsilon^c m^2$ edges that is an incomparability graph.*

Theorem 3.1 shows that string graphs and incomparability graphs are closely related. The following result shows that every dense incomparability graph contains a large balanced complete bipartite graph.

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Lemma 3.2. *(See [8].) Every incomparability graph I with m vertices and εm^2 edges contains the complete bipartite graph $K_{t,t}$ with $t \geqslant c\frac{\varepsilon}{\log 1/\varepsilon}\frac{m}{\log m}$, where c is a positive absolute constant.*

Combining Theorem 3.1 and Lemma 3.2, we have the following corollary.

Corollary 3.3. *Every string graph with m vertices and εm^2 edges contains the complete bipartite graph $K_{t,t}$ with $t \geqslant \varepsilon^{\beta}\frac{m}{\log m}$, where $\beta \geqslant 1$ is an absolute constant.*

Note that Theorem 1.4(i), the case when graph G is dense, follows immediately from Corollary 3.3 and the fact that $\text{pair-cr}(G) = \Theta(n^4)$ in this case.

We will use the following result of Kolman and Matoušek [13] which relates the pair-crossing number and the bisection width of a graph. Recall that $L_2(G)$ denotes the square root of the sum of degree squares in a graph G.

Theorem 3.4. *(See [13].) There is an absolute constant c such that if G is a graph with n vertices, then*

$$b(G) \leqslant c\log n\left(\sqrt{\text{pair-cr}(G)} + L_2(G)\right).$$

Using the same strategy as in the proof of Theorem 1.2, with Corollary 3.3 in place of Theorem 2.1 and Theorem 3.4 in place of Theorem 2.2, it is straightforward to establish Theorem 1.4.

Proof of Theorem 1.4. As already mentioned, part (i) follows directly from Corollary 3.3 and the fact that $\text{pair-cr}(G) = \Theta(n^4)$ in this case. For part (ii), let $G = (V, E)$ be a graph with n vertices and $m \geqslant 3n$ edges. Since a graph with more than $3n - 6$ edges cannot be planar, it must have a pair of crossing edges. As long as $3n \leqslant m < an$, for a fixed a, Theorem 1.4 holds as G contains a 1-grid and $1 \geqslant (m/n)^2/\log^c(m/n)$ for a sufficiently large constant $c > 0$.

Preprocess graph G and its drawing D as described in Section 2 to obtain a graph $G' = (V', E')$ with at most $2n$ vertices, m edges, and maximum degree at most $2m/n$ such that the intersection graph of E' is isomorphic to that of E. For an induced subgraph H of G', let $p(H)$ denote the number of pairs of crossing edges in H in the drawing D', let $e(H)$ be the number of edges of H and let $v(H)$ be the number of vertices of H. Corollary 3.3 implies that the intersection graph of the edges of an induced subgraph H of G' contains a bi-clique of size at least $(\frac{p(H)}{e(H)^2})^{\beta}\frac{e(H)}{\log e(H)}$, where $\beta \geqslant 1$ is the absolute constant from Corollary 3.3. This further implies Theorem 1.4 for G' (and hence for G) if

$$\frac{m^2}{n^2\log^c(m/n)} \leqslant \left(\frac{p(H)}{e(H)^2}\right)^{\beta}\frac{e(H)}{\log e(H)}, \tag{8}$$

for an absolute constant $c > 0$. Hence, it is enough to find an induced subgraph H for which

$$e(H)^{2-\frac{1}{\beta}}\log e(H)\left(\frac{m/n}{\log^{c/2}(m/n)}\right)^{\frac{2}{\beta}} < p(H), \tag{9}$$

since this already implies (8).

Next, we decompose the graph G' with Algorithm 2.3. We show that one of the induced subgraphs H in the algorithm satisfies (9), otherwise the algorithm cannot delete more than $m/2$ edges, contradicting Proposition 2.4. We use Theorem 3.4 for estimating the number of edges deleted throughout the decomposition algorithm.

Assume, to the contrary, that (9) does not hold for any induced subgraph H of G'. This gives an upper bound on $p(H)$. Note that $\text{pair-cr}(H) \leqslant p(H)$. Substituting the upper bound for $p(H)$ and using Jensen's inequality for the concave function $f(x) = x^{1-1/(2\beta)}$, we have for every $i = 0, 1, \ldots, t-1$,

$$\sum_{H\in T_i}\sqrt{\text{pair-cr}(H)} \leqslant \sum_{H\in T_i}\sqrt{p(H)} \leqslant \sum_{H\in T_i}\sqrt{e(H)^{2-\frac{1}{\beta}}\log e(H)\left(\frac{m/n}{\log^{c/2}(m/n)}\right)^{\frac{2}{\beta}}}$$
$$\leqslant O\left(\left(\frac{m/n}{\log^{c/2}(m/n)}\right)^{\frac{1}{\beta}}\sum_{H\in T_i}\log\bigl(v(H)^2\bigr)e(H)^{1-\frac{1}{2\beta}}\right)$$
$$\leqslant O\left(\left(\frac{m/n}{\log^{c/2}(m/n)}\right)^{\frac{1}{\beta}}(\log_{3/2}n-i)\sum_{H\in T_i}e(H)^{1-\frac{1}{2\beta}}\right)$$
$$\leqslant O\left(\left(\frac{m/n}{\log^{c/2}(m/n)}\right)^{\frac{1}{\beta}}(\log_{3/2}n-i)|T_i|^{\frac{1}{2\beta}}m^{1-\frac{1}{2\beta}}\right)$$
$$\leqslant O\left(\left(\frac{3}{2}\right)^{\frac{i+1}{2\beta}}(\log_{3/2}n-i)\frac{m^{1+1/(2\beta)}/n^{1/\beta}}{\log^{c/(2\beta)}(m/n)}\right).$$

From Proposition 2.5, we have

$$\sum_{H\in T_i}L_2(H) \leqslant \frac{2m}{\sqrt{n}}\sqrt{(3/2)^{i+1}}.$$

By Theorem 3.4, the total number of edges deleted during Algorithm 2.3 is

$$\sum_{i=0}^{t-1}\sum_{H\in T_i}b(H) \leqslant O\left(\sum_{i=0}^{t-1}\sum_{H\in T_i}\log v(H)\sqrt{\text{pair-cr}(H)}\right) + O\left(\sum_{i=0}^{t-1}\sum_{H\in T_i}\log v(H)\cdot L_2(H)\right)$$
$$\leqslant O\left(\frac{m^{1+1/2\beta}/n^{1/\beta}}{\log^{c/2\beta}(m/n)}\sum_{i=0}^{t-1}(\log_{3/2}n-i)^2\left(\frac{3}{2}\right)^{\frac{i+1}{2\beta}}\right) + O\left(\frac{m}{\sqrt{n}}\sum_{i=0}^{t-1}(\log_{3/2}n-i)\sqrt{\left(\frac{3}{2}\right)^{i+1}}\right)$$
$$\leqslant O\left(\frac{m^{1+1/2\beta}/n^{1/\beta}}{\log^{c/\beta}(m/n)}\left(\frac{n^2}{m}\right)^{\frac{1}{2\beta}}\log^2\left(\frac{m}{n}\right)\right) + O\left(m^{1/2}n^{1/2}\log\left(\frac{m}{n}\right)\right)$$
$$\leqslant O\left(\frac{m}{\log^{c/\beta-2}(m/n)}\right) + O\left(m\cdot\frac{\log(m/n)}{\sqrt{m/n}}\right).$$

In the third inequality, we used that $(\log_{3/2}n-i)^2 \leqslant 2(\log_{3/2}n-(t-1))^2+2((t-1)-i)^2$. Since $t-1 \geqslant \log_{3/2}2n^2/m$, we have $\log_{3/2}n-(t-1) \leqslant \log_{3/2}(m/2n)$. In the resulting upper bound on the total number of deleted edges, $O(m\log(m/n)/\sqrt{m/n}) < m/4$ if $m \geqslant an$ for a sufficiently large constant a; and $O(m\log^{2-c/\beta}(m/n)) \leqslant m/4$ if $c \geqslant 1$ is a sufficiently large constant. Hence, less than $m/2$ edges are deleted, contradicting Proposition 2.4. We conclude that an induced subgraph H of G' satisfies (9). This completes the proof of Theorem 1.4. □

4. Drawings with edges as *x*-monotone curves

It is known that Theorem 2.1 does not hold without the assumption that any two curves cross in at most a constant number of points. Using a construction from [6], Pach and G. Tóth [25] constructed for every $n\in\mathbb{N}$, a collection of n x-monotone curves whose intersection graph is dense but every biclique it contains has at most $O(n/\log n)$ vertices. Theorem 1.3 shows a stronger construction holds: the curves are edges in an x-monotone drawing of the complete bipartite graph $K_{n,n}$, where n^2 curves have only $2n$ distinct endpoints.

The proof of Theorem 1.3 builds on a crucial observation: Golumbic et al. [11] noticed a close connection between intersection graphs of curves of continuous functions defined on the interval $[0,1]$ and partially ordered sets. Consider n continuous functions $f_i:[0,1]\to\mathbb{R}$. The graph of every continuous real function is clearly an x-monotone curve. Define the partial order $\prec$ on the set of functions by $f_i \prec f_j$ if and only if $f_i(x) < f_j(x)$ for all $x \in [0,1]$. Two such x-monotone curves cross if and only if they are incomparable under this partial order $\prec$.

Lemma 4.1. *(See [11].) Let $P = \{p_1, \dots, p_n\}$ be a set of size n and $\prec$ a partial order on P. Then there is a family of continuous functions $f_1, \dots, f_n : [0,1] \to \mathbb{R}$ such that $p_i \prec p_j$ if and only if $f_i(x) < f_j(x)$ for each $x \in [0,1]$.*

Proof. Let $\Pi = \{\pi_1, \dots, \pi_t\}$ denote the collection of linear extensions $\pi_k : P \to \{1, \dots, n\}$ of the poset $(P, \prec)$. Assign to each π_k a distinct point x_k of the interval $[0,1]$, so that

$$0 = x_1 < x_2 < \cdots < x_t = 1.$$

For each $p_i \in P$, define a continuous, piecewise linear function $f_i(x)$, as follows. For any k $(1 \leqslant k \leqslant t)$, set $f_i(x_k) = \pi_k(p_i)$, and let $f_i(x)$ change linearly over the interval $[x_k, x_{k+1}]$ for $k < t$.

Obviously, whenever $p_i \prec p_j$ for some $i \neq j$, we have that $\pi_k(p_i) < \pi_k(p_j)$ for every k, and hence $f_i(x) < f_j(x)$ for all $x \in [0,1]$. On the other hand, if p_i and p_j are incomparable with respect to the ordering $\prec$, we find that there are indices k and k' $(1 \leqslant k \neq k' \leqslant t)$ such that $f_i(x_k) < f_j(x_k)$ and $f_i(x_{k'}) > f_j(x_{k'})$, therefore, by continuity, the curves of f_i and f_j must cross at least once in the interval $(x_k, x_{k'})$. This completes the proof. □

The following lemma is the key for the proof of Theorem 1.3. It presents a partially ordered set of size n^2 such that every bi-clique in its incomparability graph has size $O(n^2/\log n)$, yet it can be represented with a set of n^2 x-monotone curves having only $2n$ distinct endpoints. In the proof of Lemma 4.1, all x-monotone curves have distinct endpoints. In the proof of Theorem 1.3, we start with the same representation of a poset, but deform some x-monotone curves to have a common endpoint if they correspond to consecutive elements in some linear extension of the poset.

Lemma 4.2. *For every $n \in \mathbb{N}$, there is a partially ordered set P with n^2 elements satisfying the following properties*:

1. *every bi-clique in the incomparability graph of P has at most $O(n^2/\log n)$ nodes,*
2. *there are equitable partitions $P = P_1 \cup \cdots \cup P_n$ and $P = Q_1 \cup \cdots \cup Q_n$ such that*
   (a) *for each i, there is a linear extension of P where the elements of P_i are consecutive,*
   (b) *there is a linear extension of P where the elements of each Q_j are consecutive, and*
   (c) *for every i and j, we have $|P_i \cap Q_j| = 1$.*

We now prove Theorem 1.3, pending the proof of Lemma 4.2. The following result is a restatement of Theorem 1.3.

Theorem 4.3. *There is an x-monotone drawing of $K_{n,n}$ such that every bi-clique in the intersection graph of the edges has size at most $O(n^2/\log n)$.*

Proof. Let P be a poset described in Lemma 4.2. Represent P with x-monotone curves as in the proof of Lemma 4.1 such that the last linear extension π_t has property (b) of Lemma 4.2, that is, the elements of each Q_j are consecutive in π_t.

We transform the n^2 x-monotone curves representing P into an x-monotone drawing of $K_{n,n}$. We introduce two vertex classes, each of size n, as follows. Along the line $x = 1$, the right endpoints of the x-monotone curves in each Q_j are consecutive. Introduce a vertex on $x = 1$ for each Q_j, and make it the common right endpoint of all curves in Q_j by deforming the curves over the interval $(x_{t-1}, 1]$ but keeping their intersection graph intact. These n vertices along the line $x = 1$ form one vertex class of $K_{n,n}$.

For each i, there is a vertical line $x = x_i$ along which the x-monotone curves in P_i are consecutive. Introduce a vertex for each P_i on line $x = x_i$, and make it the common left endpoint of all curves in P_i by deforming the curves over the interval $[x_i, x_{i+1})$ and erasing their portions over the interval $[0, x_i)$.

These n vertices form the second vertex class of $K_{n,n}$. After truncating and slightly deforming the n^2 curves representing P, we have constructed an x-monotone drawing of $K_{n,n}$.

Note that the intersection graph of the edges of this drawing of $K_{n,n}$ is a subgraph of the incomparability graph of P, so every bi-clique of the intersection graph of the edges has size at most $O(n^2/\log n)$. $\square$

4.1. Proof of Lemma 4.2

It remains to prove Lemma 4.2. The construction of the poset P in Lemma 4.2 uses known constructions of constant-degree expander graphs with large girth. The *girth* of a graph is the length of the shortest cycle. A graph with v vertices and girth $\Omega(\log v)$ is said to have *large girth*. An undirected graph G is an *ε-expander* if for every vertex subset $S \subset V(G)$ with $|S| \leqslant |V(G)|/2$, we have $|N(S) \setminus S| \geqslant \varepsilon|S|$, where $N(S)$ is the set of vertices adjacent to at least one vertex in S. For a group G and symmetric subset $S \subset G$ (that is, $S = S^{-1}$) not containing the unit element of G, the *Cayley graph* $\Gamma(G, S)$ has vertex set G and (x, y) is an edge if and only if $x = ys$ for some $s \in S$. The Cayley graph $\Gamma(G, S)$ is $|S|$-regular with $|G|$ vertices. There is an integer $d \in \mathbb{N}$ and a constant $\varepsilon > 0$ for which there is an infinite family $\{\Gamma_i\}_{i\geqslant 1}$ of d-regular Cayley graphs with $\lim_{i\to\infty} \frac{|V(\Gamma_{i+1})|}{|V(\Gamma_i)|} = 1$ such that each Γ_i is an ε-expander with large girth. See [16] or [2] for explicit constructions.

We next introduce analogous terminology for directed graphs (digraphs). A digraph is d-regular if the in-degree and out-degree of every vertex is d. For a subset S of vertices in a digraph D, let $N_+(S)$ denote the set of vertices $x \in V(D)$ for which there is a vertex $s \in S$ such that (s, x) is an edge of D. Similarly, $N_-(S)$ is the set of vertices $y \in V(D)$ for which there is a vertex $s \in S$ such that (y, s) is an edge of D. A digraph D has *path-girth* k if k is the smallest positive integer such that some pair of vertices is connected by two distinct walks of length k. Equivalently, denoting by A_D the adjacency matrix of D, it has path-girth k if $A_D^1, \dots, A_D^{k-1}$ are all 0–1 matrices, but the matrix A_D^k has an entry greater than 1. A digraph D is an *ε-expander* if both $N_+(S) \setminus S$ and $N_-(S) \setminus S$ have size at least $\varepsilon|S|$ for all $S \subset V(D)$ with $|S| \leqslant |V(D)|/2$. For a group G and subset $S \subset G$, the *Cayley digraph* $D(G, S)$ has vertex set G and (x, y) is an edge if and only if $y = xs$ for some $s \in S$.

Assume that the Cayley graph $\Gamma(G, S)$ is a d-regular ε-expander with girth g. Construct an asymmetric set $S' \subset S$ of size $|S'| = |S|/2 = d/2$ by deleting an arbitrary element from each pair $\{s, s^{-1}\}$ in the symmetric set S. The Cayley digraph $D(G, S')$ is $(d/2)$-regular with path-girth at least $g/2$. Furthermore, for any subset $T \subset G$, since the orbit of every element $s \in S'$ is a cycle, these cycles provide a bijection between edges directed out from T and edges directed into T in $D(G, S')$. It follows that $D(G, S')$ is an $(\varepsilon/|S|)$-expander. From the construction in [16] of Cayley graphs that are constant-degree expanders with large girth, we can conclude the following.

Lemma 4.4. *There is a positive integer d and constants $c, \varepsilon > 0$ for which there is an infinite family $\{D_i\}_{i\geqslant 1}$ of d-regular Cayley digraphs with $\lim_{i\to\infty} \frac{|V(D_{i+1})|}{|V(D_i)|} = 1$ such that each D_i is an ε-expander with path-girth greater than $c \log |D_i|$.*

Let $D = D(G, S)$ be a Cayley digraph as in Lemma 4.4 with $|D| = v$. For every $a \in \mathbb{N}$, we define a poset $P(a, D)$ with ground set $G \times \{1, 2, \dots, a\}$, generated by the relations $(j_1, k_1) \prec (j_2, k_2)$ whenever $k_2 = k_1 + 1$ and (j_1, j_2) is an edge of D.

Let $P_0 = P(a, D)$ with $a = \lfloor \min(c, (10 \log d)^{-1}) \cdot \log v \rfloor$. It is clear that P_0 has $a|D| = \Theta(v \log v)$ elements. Partition P_0 into subsets $P_0 = X_1 \cup \dots \cup X_a$, where $X_k = \{(j, k) \in P_0\colon\ j \in G\}$. Let $X_{\leqslant k} = \bigcup_{i=1}^k X_i$ and similarly $X_{\geqslant k} = \bigcup_{i=k}^a X_i$ for k, $1 \leqslant k \leqslant a$.

Proposition 4.5. *The poset in Fig.* 1(a) *is not a subposet of P_0.*

Proof. Each path down the Hasse diagram of P_0 between comparable elements $y \prec x$ corresponds to a walk in the digraph D. Since the path-girth of D is greater than a, there is a unique path in the Hasse diagram between y and x. if $x, y, z, w \in P_0$ satisfy both $y \prec z \prec x$ and $y \prec w \prec x$, then z and w must be in this unique path and so they are comparable. $\square$

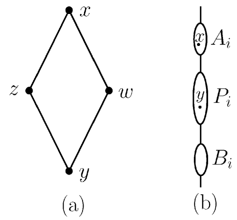

Fig. 1. (a) The Hasse diagram of a four element excluded subposet of P_0. (b) A linear extension of P where $B_i \prec P_i \prec A_i$.

We show, by essentially the same argument as in [6] (proof of Lemma 9), that the incomparability graph of the partially ordered set P_0 does not contain large bi-cliques.

Proposition 4.6. *The size of every bi-clique in the incomparability graph of P_0 is $O(|D|) = O(v)$.*

Proof. Assume that P_0 contains two disjoint subsets, A and B, each of size m, such that every element of A is incomparable with every element of B. It is enough to show that $m = O(v)$.

First we show that there is an integer κ, $1 \leqslant \kappa \leqslant a$, and two subsets $A' \subseteq X_{\geqslant\kappa}$, $B' \subseteq X_{\leqslant\kappa}$ such that $|A'| \geqslant m/2$, $|B'| \geqslant m/2$ and every element of A' is incomparable with every element of B'. Let κ, $1 \leqslant \kappa \leqslant a$, be the integer such that $|A \cap X_{\leqslant\kappa}| \geqslant |A|/2 > |A \cap X_{\leqslant(\kappa-1)}|$. That is, we have both $|A \cap X_{\geqslant\kappa}| \geqslant |A|/2$ and $|A \cap X_{\leqslant\kappa}| \geqslant |A|/2$. If $|B \cap X_{\leqslant\kappa}| \geqslant m/2$, then let $A' = A \cap X_{\geqslant\kappa}$ and $B' = B \cap X_{\leqslant\kappa}$, otherwise let $A' = B \cap X_{\geqslant\kappa}$ and $B' = A \cap X_{\leqslant\kappa}$.

Let $\widehat{A} = \{x \in X_\kappa\colon\ \exists y \in A' \text{ with } x \preccurlyeq y\}$ be the set of all elements in X_κ that are less than or equal to some element in A'. Similarly, let $\widehat{B} = \{x \in X_\kappa\colon\ \exists y \in B' \text{ with } y \preccurlyeq x\}$. Note that $\widehat{A}$ and $\widehat{B}$ are disjoint, otherwise $x \in \widehat{A} \cap \widehat{B}$ would imply that $y_1 \preccurlyeq x \preccurlyeq y_2$ for some elements $y_1 \in B'$ and $y_2 \in A'$. Assume that $|\widehat{A} \cap X_\kappa| \leqslant |X_\kappa|/2$, where $|X_\kappa| = v$. The case that $|\widehat{B} \cap X_\kappa| \leqslant |X_\kappa|/2$ is analogous. Observe that for every $i = 0, 1, \ldots, a-\kappa$, we have

$$|A' \cap X_{\kappa+i}| \leqslant |\widehat{A} \cap X_\kappa| \left(\frac{1}{1+\varepsilon}\right)^i \leqslant \frac{v}{2}\left(\frac{1}{1+\varepsilon}\right)^i,$$

since D is an ε-expander and so $|\widehat{A} \cap X_\kappa| \geqslant \min(v/2, |A' \cap X_{\kappa+i}|(1+\varepsilon)^i|)$. Therefore, we can estimate the size of A' from both sides as

$$\frac{m}{2} \leqslant |A'| \leqslant \sum_{i=0}^{a-\kappa} \frac{v}{2}\left(\frac{1}{1+\varepsilon}\right)^i < \frac{v}{2}\sum_{i=0}^{\infty}\left(\frac{1}{1+\varepsilon}\right)^i = \frac{1+\varepsilon}{2\varepsilon} v,$$

which gives $m \leqslant \frac{1+\varepsilon}{\varepsilon} v$, as required. $\square$

Having finished the necessary preparation, we are now ready to prove Lemma 4.2.

Proof of Lemma 4.2. The poset P required for Lemma 4.2 will be a linear-sized subposet of P_0. We next describe the construction of P.

A *chain* is a set of pairwise comparable elements. The maximum chains in P_0 each have size a, having one element from each of $X \times \{k\}$, $k = 1, 2, \ldots, a$. Fix an element $t \in S$. For each $g \in G$, consider the chain $C_g = \{(gt^{k-1}, k)\colon\ 1 \leqslant k \leqslant a\}$. The collection $\mathcal{C} = \{C_g\colon\ g \in G\}$ consists of disjoint chains, so $\mathcal{C}$ is a partition of the poset P_0, and the collection $\mathcal{C}$ has cardinality v.

We successively choose disjoint subsets $P_1, \ldots, P_{ha}$ of P_0, each of which is the union of $h = \Theta(\sqrt{v/\log v})$ chains of $\mathcal{C}$. The number of subsets is $ha = \Theta(\sqrt{v \log v})$, each containing $ha = \Theta(\sqrt{v \log v})$ elements of P_0. Each P_i, $1 \leqslant i \leqslant ha$, has the property that any two comparable elements in P_i belong to the same chain of $\mathcal{C}$. We can choose the h chains of each P_i greedily: after choosing the ℓth chain in P_i, we have to choose the $(\ell+1)$st chain such that none of its elements are comparable with any element of the first ℓ chains of P_i. It is easy to see that each element of P_0

is comparable with at most $d + d^2 + \cdots + d^{a-1} < d^a \leqslant v^{1/10}$ other elements of P_0 (see Lemma 8 of [6]). Since at most $\ell a v^{1/10} \leqslant h a v^{1/10} = v^{3/5+o(1)}$ of the $v - (i-1)h - \ell = \Theta(v)$ remaining chains contain an element comparable with the first ℓ chains of P_i, almost any of the remaining chains can be chosen as the $(\ell+1)$th chain of P_i. Finally, let $P = P_1 \cup \cdots \cup P_{ha}$. As mentioned earlier, we have $|P| = \Theta(v \log v) = \Theta(|P_0|)$, and the largest bi-clique in the incomparability graph of P is of size $O(|P_0|/\log|P_0|) = O(|P|/\log|P|) = \Theta(v)$.

If any element of $P \setminus P_i$ is both greater than an element of P_i and less than another element of P_i, then these two elements of P_i are comparable. By construction, if two elements of P_i are comparable, then they belong to the same chain. Since the poset in Fig. 1(a) is not a subposet of P_0, no element of $P_0 \setminus C_g$, $C_g \in \mathcal{C}$, can be both greater than an element of C_g and less than another element of C_g. Therefore, no element of $P \setminus P_i$ can be both greater than an element of P_i and less than another element of P_i.

Consider the partition $P = A_i \cup P_i \cup B_i$, where an element $x \in P \setminus P_i$ is in A_i if and only if there is an element $y \in P_i$ such that $y \prec x$. There is a linear extension of P in which the elements of A_i are the largest, followed by the elements of P_i, and the elements of B_i are the smallest (see Fig. 1(b)). Indeed, any linear extension of the poset restricted to A_i, followed by any linear extension of the poset restricted to P_i, followed by any linear extension of the poset restricted to B_i will do. This is because no element of $P \setminus P_i$ can be both greater than an element of P_i and less than another element of P_i.

Note that each X_k contains exactly $h^2 a$ elements in P, h elements from each P_i. Arbitrarily partition each X_k into h sets $X_k = Q_{(k-1)h+1} \cup \cdots \cup Q_{kh}$ such that each Q_j contains one element from each P_i. Since the elements in each X_k form an *antichain* (a set of pairwise incomparable elements), any linear order of the elements of P for which the elements of X_k are smaller than the elements of X_ℓ for $1 \leqslant k < \ell \leqslant a$ is a linear extension of P. Hence, there is a linear extension of P such that, for each j, the elements of every Q_j are consecutive.

We have established that P has all the desired properties. We can choose v such that $n \leqslant ha$ and $ha = O(n)$, so $v = \Theta(n^2/\log n)$. If ha is not exactly n, we may simply take the subposet whose elements are $(P_1 \cup \cdots \cup P_n) \cap (Q_1 \cup \cdots \cup Q_n)$. This completes the proof of Lemma 4.2. □

5. Monotone properties

If a graph is drawn with at most k crossings between any two edges and the graph has some additional property, then the lower bound for the largest ℓ-grid in Theorem 1.2 can be improved.

A graph property $\mathcal{P}$ is *monotone* if whenever a graph G satisfies $\mathcal{P}$, every subgraph of G also satisfies $\mathcal{P}$, and whenever graphs G_1 and G_2 satisfy $\mathcal{P}$, then their disjoint union also satisfies $\mathcal{P}$. The *extremal number* $\mathrm{ex}(n, \mathcal{P})$ denotes the maximum number of edges that a graph with property $\mathcal{P}$ on n vertices can have. For graphs satisfying a monotone graph property, the bound (1) of the Crossing Lemma can be improved [22]. In particular, if $\mathcal{P}$ is a monotone graph property and $\mathrm{ex}(n, \mathcal{P}) = O(n^{1+\alpha})$ for some $\alpha > 0$, then there exist constants $c, c' > 0$ such that for every graph G with n vertices and $m \geqslant cn \log^2 n$ edges that satisfies property $\mathcal{P}$, the crossing number is at least $\mathrm{cr}(G) \geqslant c' m^{2+1/\alpha}/n^{1+1/\alpha}$. Furthermore, if $\mathrm{ex}(n, \mathcal{P}) = \Theta(n^{1+\alpha})$, then this bound is tight up to a constant factor. One can prove the following strengthening of Theorem 1.2.

Theorem 5.1. *Let $\mathcal{P}$ be a monotone graph property such that* $\mathrm{ex}(n, \mathcal{P}) = O(n^{1+\alpha})$ *for some* $\alpha > 0$. *For every $k \in \mathbb{N}$, there exist positive constants c and c_k such that every drawing of a graph $G = (V, E)$ satisfying property $\mathcal{P}$, having n vertices and $m \geqslant cn \log^2 n$ edges in which no two edges cross in more than k points contains an ℓ-grid with $\ell \geqslant c_k (m/n)^{1+1/\alpha}$.*

Similarly, Theorem 1.4 can be strengthened for graphs satisfying a monotone graph property.

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